YOUR NAME
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Homework 7

1. (1 point) Suppose that X is a discrete random variable, with cdf given by Table 1. Find $E[X]$ and $\text{Var}(X)$.

x	$F(x)$
0	0
1	.1
2	.3
3	.7
4	.8
5	1.0

Table 1: The cdf for X in Problem 1

2. (1 point) Suppose that X follows a Poisson(λ) distribution. Find $E\big[1/(X+1)\big]$.

3. (1 point) Let X have a Gamma(α, λ) distribution. For those values of α and λ for which it is defined, find $E[1/X]$.

4. (1 point) Suppose we have n independent and identical samples from a population, denoted $X_1, X_2, \ldots, X_n$. Let $E[X_i] = \mu$ (which we assume exists, and is the same for all i).

   (a) Calculate the expected value of $\bar{X}_n$, the sample mean

   $$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

   $E\left[\bar{X}_n\right] \rightarrow$ linearity of expectation.

   (b) Is $\bar{X}_n$ an *unbiased* estimator of μ? If not, find values of a and b such that the linear transformation such that $a\bar{X}_n + b$ is an unbiased estimate of μ.

5. (2 point) Using the same setup as above, suppose now that the variance of the population is also finite, i.e., $\text{Var}(X_i) = \sigma^2 < \infty$.

   $\text{Var}(X) = E\left[(X-\mu)^2\right]$

   (a) Consider calculating the average squared deviance Y

   $$\text{biased} \rightarrow Y = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

   $\sigma^2, \quad \hat{\sigma}^2 = \frac{1}{n-1}\sum (X_i - \bar{X}_n)^2$

   What is $E[Y]$?

   (b) Is Y an *unbiased* estimator of σ^2? If not, find values of a and b such that the linear transformation $aY + b$ is an unbiased estimate of σ^2.

6. (1 points) Let X be an exponential(λ) random variable, which has standard deviation $\sigma = 1/\lambda$. Find

   $$P\big(|X - E[X]| > k\sigma\big),$$

   for $k = 2, 3, 4$, and compare the results to the bounds from Chebyshev's inequality.

7. (2 points) Suppose a child randomly types the letters Q W E R T Y on a keyboard 1000 times.

   (a) What is the expected number of times that the sequence QQQQ appears (counting overlaps)?

   (b) Can you provide an upper bound on the probability that the sequence QQQQ appears more than n times? How many times n for it to happen would you be surprised?

X is only positive.

$P(X < -1) = 0$

$\boxed{5b}$ $Y = \frac{1}{n}\sum(x_i - \bar{x}_n)^2$

Hard

$E[Y] = E\left[\frac{1}{n}\sum(x_i - \bar{x})^2\right]$

$= \frac{1}{n}\sum_{i=1}^{n} E\left[(x_i - \bar{x})^2\right]$

$E[X_i \bar{X}_n] \neq E[X_i]E[\bar{X}_n]$

$X_i(x_1 + x_2 + \ldots + x_i + \ldots + x_n)$

Easy

$E[Y] = E\left[\frac{1}{n}\sum(x_i - \bar{x})^2\right]$

$= E\left[\frac{1}{n}\sum(x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right]$

$= E\left[\frac{1}{n}\left(\sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2\right)\right]$

$\bar{x} = \frac{1}{n}\sum x_i$

$n\bar{x} = \sum x_i$

$-2\bar{x}(n\bar{x})$

$E[\bar{x}]$