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Homework 8

1. (1 point) Suppose a turtle lays a random number of eggs Y, and the probability that a hatchling from any given egg survives is p. If Y is modeled using a Poisson(λ) distribution, and the ith egg survives following an independent model $X_i \sim \text{Bernoulli}(p)$ what is the expected value and variance of X, the total number of hatchlings that survive?

2. (1 point) Show that $E\big[\text{Var}(Y|X)\big] \leq \text{Var}(Y)$.

3. (1 point) Let $T \sim \text{Exp}(\lambda)$, and let U be uniform on the interval $[0, T]$ (that is, we have a hierarchical model where U depends on T). Find $E[U]$ and $\text{Var}(U)$.

4. Use the definition of Covariance to show the following:
   (a) (0.5 points) $\text{Cov}(aX, bY) = ab\,\text{Cov}(X, Y)$.
   (b) (0.5 poitns) $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$.
   (c) (1 point) Now combine the results from parts 4a and 4b to show the following:

$$\text{Cov}(aW + bX, cY + dZ) = ac\,\text{Cov}(W, Y) + bc\,\text{Cov}(X, Y) + ad\,\text{Cov}(W, Z) + bd\,\text{Cov}(X, Z).$$

5. (1 point) Consider a symmetric random walk, starting at the position $S_0 = 0$. At time n, we write the position S_n as

$$S_n = S_{n-1} + X_n,$$

   where the X_n are mean zero, independent steps with variance σ^2. For any $n, m > 0$, find an expression for $\text{Cov}(S_n, S_m)$.

6. (3 points) In time series analysis, we are interested in the behaviour of a sequence of random variables $X_1, X_2, \ldots$, where the order of the random variables is an important feature of their behavior. An important concept in time series is *stationarity*. A times series is said to be (*weakly*) *stationary* if $\text{Var}(X_n) < \infty$ for all n, and

   - $E[X_n] = \mu < \infty$, for all $n \in \{1, 2, \ldots\}$ (the mean doesn't depend on time).
   - $\text{Cov}(X_i, X_j) = \gamma_{|i-j|}$, or that the covariance only depends on the distance between i and j. In other words, for all time points n and values $h \in \{0, 1, 2, \ldots\}$, there exists some γ_h such that

$$\text{Cov}(X_n, X_{n+h}) = \text{Cov}(X_{n+h}, X_n) = \gamma_h. \tag{1}$$

   For this problem, we will assume that X_n comes from an Auto-regressive (1) model. That is, for some $-1 < \phi < 1$, we assume that

$$X_n = \phi X_{n-1} + \epsilon_n, \tag{2}$$

   where ϵ_n are iid $N(0, \sigma^2)$ random variables. We will now assume that X_0 is sampled from the *stationary distribution*, or that $E[X_0] = 0$, and $\text{Var}(X_0) = \gamma_0$. **Your task is to use the information above to find an expression for γ_h in terms of** h, σ^2 and ϕ. The steps below will should help you through this process.

   (a) (1 point) Step (1): Use properties of the covariance, Equations 1 and 2 to express γ_h in terms of γ_{h-1}:

$$\begin{aligned}\gamma_h &= \text{Cov}(X_n, X_{n+h}) \quad \text{(By Eq. 1)} \\ &= \text{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}) \quad \text{(By Eq. 2)} \\ &= \ldots \text{ You continue from here } \ldots\end{aligned}$$

(b) (1 point) Step (2): Using your result in the previous step, get an expression for γ_h in terms of γ_0.

(c) (1 point) Step (3): Now our goal is to get an expression for γ_0, which is $\mathrm{Cov}(X_n, X_n)$ for all n:

$$\begin{aligned}\gamma_0 &= \mathrm{Cov}(X_n, X_n) \\ &= \mathrm{Cov}(\phi X_{n-1} + \epsilon_{n-1}, \phi X_{n-1} + \epsilon_{n-1}) \quad \text{(By Eq. 2)} \\ &= \ldots \text{ You continue from here } \ldots\end{aligned}$$

(d) (0 points, trivial) Step (4): Combine your results from Step 2 and Step 3 to get an expression of γ_h in terms of h, σ^2 and ϕ.

(e) (BONUS, +0.5): How does your result change if X_0 is not from the stationary distribution? Is the process weakly stationary? For instance, suppose $X_0 = x_0$, so that X_0 is not random but some constant.