YOUR NAME
YOUR EMAIL
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Homework 9 (Chapter 5)

1. (1 point) Find the moment generating function for the density $f(x) = 2x/c^2, 0 < x < c$.

2. (1 point) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E(X_i) = \mu$, and $\text{Var}(X_i) = \sigma_i^2$. Show that if $n^{-2} \sum_i \sigma_i^2 \to 0$, then $\bar{X} \xrightarrow{p} \mu$ (This is a statement in convergence in probability. Check Chapter 5 notes for a definition, as well as the examples for how you might solve this problem).

3. (1 point) Let $X_1, X_2, \ldots$ be as in the problem above, but with $E(X_i) = \mu_i$, and $n^{-1} \sum_{i=1}^{n} \mu_i \to \mu$. Show that $\bar{X} \xrightarrow{p} \mu$. **Hint:** you might need to use the triangle inequality.

4. Let X follow a Binomial(n, p) distribution.

   (a) (1 point) Using the Binomial Theorem (see, for instance, Proposition 1.2 slides from Chapter 1), derive the MGF of X.

   (b) (1 points) Using moment-generating functions, show that as $n \to \infty$, $p \to 0$, and $np \to \lambda$, the binomial distribution with parameters n and p tends to the Poisson distribution.

5. (1 point) Using moment-generating functions, show that as $\alpha \to \infty$, the gamma distribution with parameters α and λ, properly standardized, tends to the standard normal distribution. (**Hint**: we already calculated the MGF of a Gamma distribution in class. No need to rederive it here.)

6. (1 point) Suppose that $X_1, \ldots, X_{20}$ are independent random variables with density functions

$$f(x) = 2x, \quad 0 \leq x \leq 1.$$

   Let $S = X_1 + \ldots + X_{20}$. Use the central limit theorem to approximate $P(S \leq 10)$.

7. (2 points) Suppose that a measurement X_i has mean μ and variance $\sigma^2 = 25$. Let $\bar{X}$ be the average of n such independent measurements. How large should n be so that $P\big(|\bar{X} - \mu| < 1\big) = 0.95$?

   - Note that there are various approaches to this problem. Two we have covered in this class, namely using the CLT to approximate the distribution, or using Chebyshev's inequality. Try both approaches, and comment on the results.