Midterm Review

6 problems → sub. 75 min 12.5 min.

- pdf : $\boxed{f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}$ ←

Things you should know

- Probability axioms.

  Ω : sample space

  - $P(\Omega) = 1$
  - $A \subset \Omega$, $P(A) \geq 0$ $\quad A_i \subset \Omega$
  - $A_1, A_2, \ldots, A_n$ are disjoint sets, then
    - $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$
    - $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

- $P(A^c) = 1 - P(A)$

  proof: By (1), $P(\Omega) = 1$ ☆

  by def. $A \cup A^c = \Omega$

  by def. A and A^c ☆ are disjoint

  ~~by (3)~~ $P(\Omega) = 1$

  $P(A \cup A^c) = 1$

  by (3) $P(A) + P(A^c) = 1 \Rightarrow P(A^c) = 1 - P(A)$

- def'n disjoint,
- def'n independence
- $P(A\cap B) \stackrel{?}{=} P(B)P(A) \rightarrow$ Ind.
- $P(A\cup B) = P(A) + P(B) - \underbrace{P(A\cap B)}_{0}$
  $= P(A) + P(B)$ if disjoint,
- Conditional probs: $P(A|B) \begin{cases} = \dfrac{P(A\cap B)}{P(B)}, & P(B)\neq 0 \\ = 0, & \text{if } P(B)=0 \end{cases}$
- Total prob:

$$P(A) = \sum_i P(A|B_i)P(B_i), \quad B_i \text{ is a partition.}$$

$$= \underline{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

$$\bigcup_i B_i = \Omega$$

$$B_i \cap B_j = \emptyset$$

- Bayes Theorem. $P(A|B) \longmapsto P(B|A)$

$$P(B_i|A) = \frac{P(A\cap B)}{P(A)}$$

$$= \frac{P(A|B_i)P(B_i)}{\sum P(A|B_i)P(B_i)} \quad \text{Bayes' Theorem}$$

- What is a R.V.

Ω, $X: \Omega \to \mathbb{R}$, $X(\omega)$ $\{H, T\}$

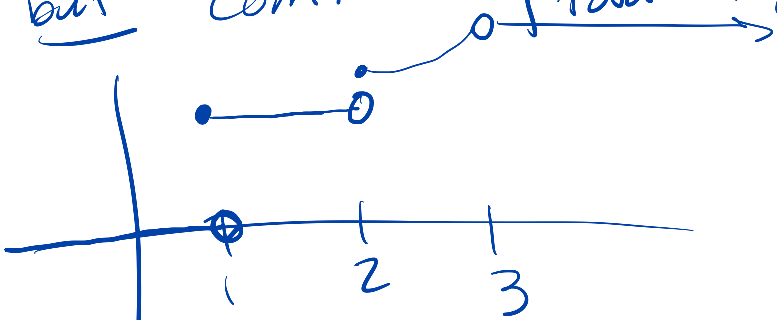
$X = \#$ heads

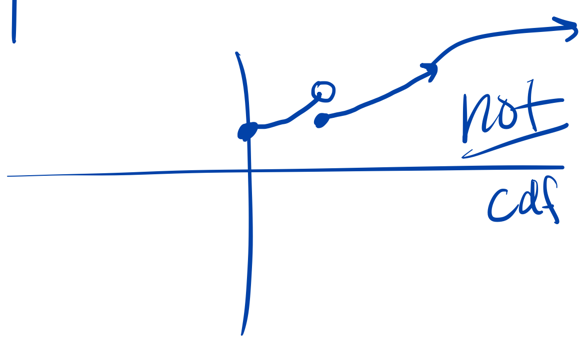
$X(\{H\}) = 1$

- random number, determined by the out come of random experiment

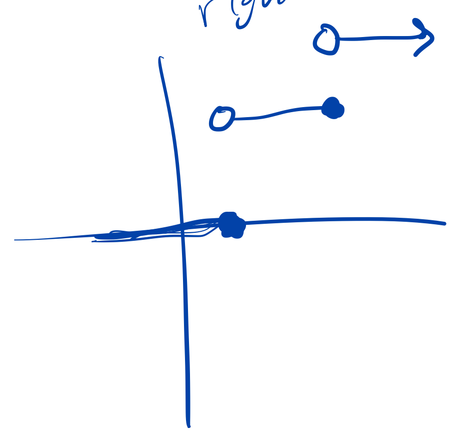
X

- discrete vs con. RV
- pdf vs pmf
- cdf : X RV, $F_X(x) = P(X \le x)$
  - $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$
  - non-decreasing, i.e., (forall $u > v$, $F_X(u) \ge F_X(v)$)
  - Right-continuous, can have jumps but continuous from right.





not right-cont.



- calc using pdf pmf.

$$P(X \in A) = \int_A f_X(x)\, dx \qquad X \sim N(\mu, \sigma^2)$$

$$A = (-\infty, -3) \cup (1, 5)$$

$$P(X \in A) = \int_{(-\infty,-3)\cup(1,5)} f_X(x)\, dx$$

$$= \int_{-\infty}^{-3} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx + \int_1^5 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

for discrete, $X \in \{1, 2, 3, 4, \dots\}$

for all $\omega \in \Omega$ $X(\omega)$ is unique

A_i is the event $X(\omega) = x_i$, $x_i \in \{1, 2, \dots\}$

disjoint.

$$P(X \in \{x_1, x_2, x_3\}) = \sum_{x_i} P(X = x_i) = \sum_{x_i} p(x_i)$$

$$P(X \geq a) = \sum_{i=a}^{\infty} p(i) = 1 - \sum_{i=0}^{a-1} p(i)$$

- Basic counting skills,
  - → $n!$ ←
  - → "choose" $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$
  - → permutations, etc.

- $\sum_{x} p(x) = 1$, $\int_{x} f(x)\,dx = 1$

- $\mathbb{1}(a \le x \le b) = \begin{cases} 1 & \text{if } a \le x \le b \\ 0 & \text{o.w.} \end{cases}$

  $f(x)$

- "cdf method" X, $Y = g(X)$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$
$$= \frac{d}{dy} P(Y \le y)$$
$$= \frac{d}{dy} P(g(X) \le y)$$
$$= \frac{d}{dy} P(X \le g^{-1}(y))$$
$$= \frac{d}{dy} \left(F_X(g^{-1}(y)) \right)$$
$$= f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

- joint vs marg. vs cond. distn.

(X, Y) jointly discrete, with pmf $p(x,y)$

$$p_X(x) = \sum_y p(x,y)$$

$$= \sum_y p(x|y)\, p(y)$$

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$$

$(X, Y) \to$ cont. $f(x,y)$

$$f_X(x) = \int f(x,y)\,dy$$

$$= \int \underbrace{f_{X|Y}(x|y)}\, f_Y(y)\, dy$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} \rightarrow \text{not zero} \\ 0 \quad \text{if } f_Y(y) = 0 \end{cases}$$

Ch3, slide 63

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda y} \quad 0 \le x \le y$$

→ find $f_X(x)$, $f_Y(y)$

$$f_X(x) = \int_y f_{X,Y}(x,y)\,dy$$

$$= \int_x^\infty \lambda^2 e^{-\lambda y}\,dy$$

$$= \lambda^2 \int_x^\infty e^{-\lambda y}\,dy$$

$$= \lambda^2 \left[-\frac{1}{\lambda} e^{-\lambda y}\right]_x^\infty$$

$=$

$$= \lambda^2 \left[\frac{1}{\lambda} e^{-\lambda x} \right]$$

$$= \boxed{\lambda e^{-\lambda x}, \quad x \geq 0}$$

exp.

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{\lambda^2 e^{-\lambda y} \, 1[0 \leq x \leq y]}{\lambda e^{-\lambda x} \, 1[0 \leq x]}$$

$$= \lambda e^{-\lambda y + \lambda x} \, 1[x \leq y]$$

$$= \lambda e^{-\lambda(y-x)} \, 1[x \leq y]$$

$$f_X(x) = \int_Y f_{X,Y}(x,y)\,dy$$

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x \le y \\ 0 & \text{o.w} \end{cases}$$

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

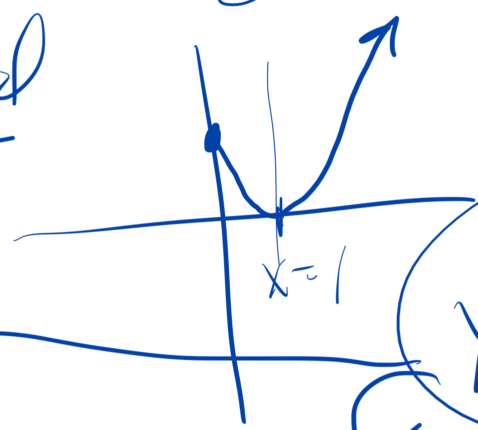
$$1[0 \le x \le y]$$

$$1[0 \le x] \cdot 1[x \le y]$$

HW5 p3: $X \sim \exp(1)$

not monotonic $\rightarrow$ $Y = (X-1)^2$

• cdf method



$f_Y(y)$

$y = (x-1)^2$

$$P(Y \le y) = P\left((X-1)^2 \le y\right)$$

$$= P\left(-\sqrt{y} \le (X-1) \le \sqrt{y}\right)$$

for $X \le 0$, $P(X \le 0) = 0$

$$= P\left(1-\sqrt{y} \le X \le 1+\sqrt{y}\right)$$

$1-\sqrt{y} < 0$

$$= \int_{1-\sqrt{y}}^{1+\sqrt{y}} f_X(x)\,dx$$

for $x \le 0$, $f(x) = 0$

$y \le 0$: $(x-1)^2 = Y \rightarrow f_Y(y) = 0$

$0 \le y \le 1$: $\begin{cases} 1+\sqrt{y} \\ 1-\sqrt{y} \end{cases}$ $\int_{1-\sqrt{y}}^{1+\sqrt{y}} f_X(x)\,dx$

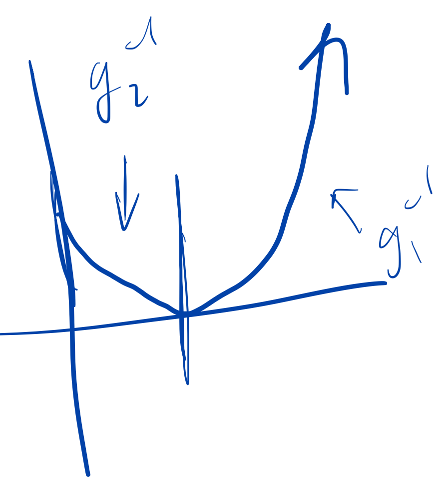
$y \ge 1$: $1-\sqrt{y} < 0$ $\int_{1-\sqrt{y}}^{1+\sqrt{y}} f_X(x)\,dx$, $1-\sqrt{y} < 0$

$$= \int_0^{1+\sqrt{y}} f_X(x)\,dx$$

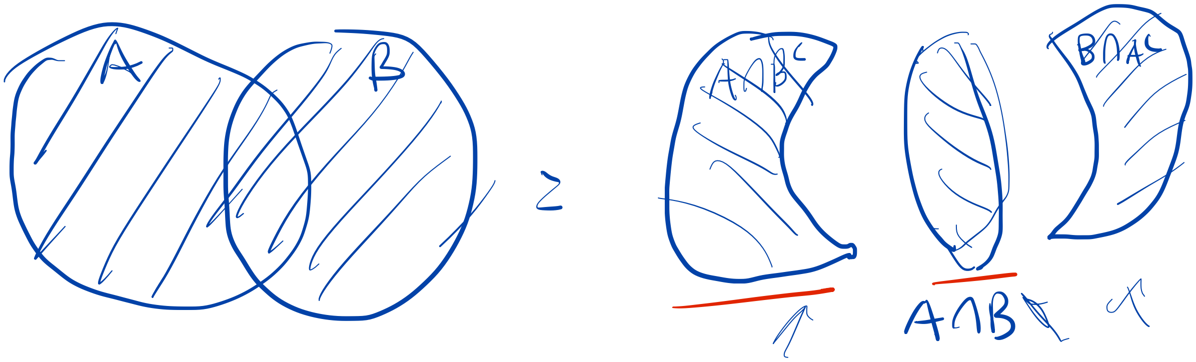
$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$g_1^{-1}(y) = 1+\sqrt{y}$$

$$g_2^{-1}(y) = 1-\sqrt{y}$$



- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



- disjoint sets, we get to <u>sum</u> prob.

$$A \cup B = \overset{(1)}{(A \cap B^c)} \cup \overset{(2)}{(A \cap B)} \cup \overset{(3)}{(B \cap A^c)}$$

$$P(A \cup B) = P\big((1) \cup (2) \cup (3)\big)$$

$$= P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$$

$A = (A \cap B^c) \cup (A \cap B)$

$P(A) = P(A \cap B^c) + P(A \cap B)$

$P(B) = P(A \cap B) + P(B \cap A^c)$

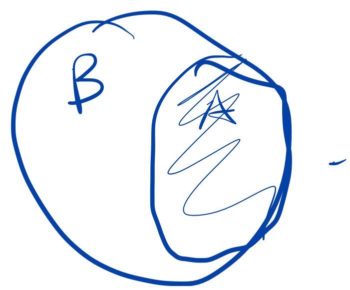
$$P(A) + P(B) = 2P(A \cap B) + P(A \cap B^c) + P(B \cap A^c)$$

$$P(A)+P(B)-P(A\cap B) = P(A\cap B)+P(A\cap B^c)+P(B\cap A^c)$$

$$= \boxed{P(A\cup B)}$$

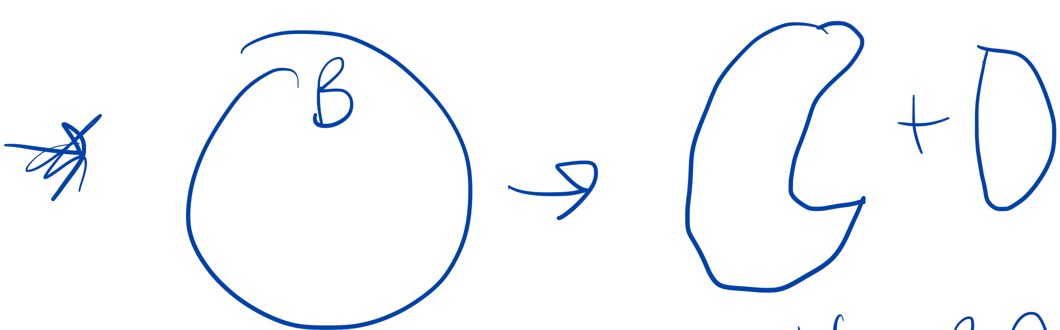
$A \subset B$, $P(A) \le P(B)$

→ proof:



→ what do we know
disjoint sets, add prob.

$A \subset B$, $A \cap B = A$



$B\cap A^c$ $B\cap A$

$$B = (B \cap A^c) \cup (B \cap A)$$

$$= (B \cap A^c) \cup A$$

$$P(B) = \underbrace{P(B \cap A^c)}_{\geq 0} + P(A)$$

$$\geq 0 + P(A)$$

$$P(B) \geq P(A) \quad \square$$

- symmetric arguments

$$f(x,y) = \frac{3}{32}\left(x^2+y^2+2\right)$$

$$-1 \le x \le 1$$
$$-1 \le y \le 1$$

$$f(x,y) = f(y,x)$$

find pdf of X and pdf Y.

$$f_X(x) = \int f(x,y)\,dy$$

$$f_Y(y) = \ldots \quad \text{by symmetry}$$

$$\left.\begin{array}{l} X \mid Y \\ Y \mid X \end{array}\right\}$$