YOUR NAME
YOUR EMAIL
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Homework 5

1. (4 points) Beta-Binomial Conjugacy.
   (a) (3 points) Show that the Beta family of distributions is conjugate to the Binomial family of distributions (i.e., if the likelihood is Binomial(n, p), then if the prior is $P \sim \text{Beta}(\alpha, \beta)$, then the posterior $P|X$ is a beta-distribution).
   (b) (1 point) Use the result above to argue why a Uniform$(0, 1)$ prior for P results in a Beta-distriubted posterior.
2. (6 points) Approximate the following integrals using some form of Monte-Carlo. Describe the distribution used and the overall approach you picked.
   (a) (2 points) $\int_0^1 \sin\left(e^{-x}\right)x^2\,dx$
   (b) (2 points) $\int_0^2 e^{x^2}\,dx$
   (c) (2 points) $\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx$
3. (6 points) In class, we calculated the posterior mean $\Lambda|X$, where $X \sim \text{Pois}(\lambda)$, and the prior distribution was $\Lambda \sim U(0, 100)$. This required calculating the following integral:

$$E[\Lambda|X = x^*] = \frac{\int_0^{100} \tau \cdot \tau^{\sum_i x_i^*} e^{-n\tau}\,d\tau}{\int_0^{100} \tau^{\sum_i x_i^*} e^{-n\tau}\,d\tau}$$

   (recall the denominator is just a normalizing constant.) To perform this calculation, it required making some transformations to avoid numeric overflow / underflow issues. In particular, we showed

$$E[\Lambda|X = x^*] = \frac{\int_0^{100} \tau \cdot \tau^{\sum_i x_i^*} e^{-n\tau}\,d\tau}{\int_0^{100} \tau^{\sum_i x_i^*} e^{-n\tau}\,d\tau} = \frac{\int_0^{100} \lambda \exp\left\{g(\lambda) - g(\bar{x}^*)\right\} d\lambda}{\int_0^{100} \exp\left\{g(\lambda) - g(\bar{x}^*)\right\} d\lambda},$$

   where $g(\lambda) = \sum_i x_i^* \log \lambda - n\lambda$.
   (a) (3 points) In class, we used the `integrate` function in R to estimate the posterior mean. **Now, estimate the posterior mean using a Monte-Carlo method.** (The numerically-stable version will probably be easier).
   (b) (3 points) Approximate the posterior variance $\text{Var}(\Lambda|X = x^*)$ using Monte-Carlo (HINT: try finding $E[\Lambda^2|X = x^*]$).