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Homework 9

1. Recall the definition of the likelihood ratio:

$$\lambda(x) = \frac{\sup_{\theta \in \Theta_0} L(\theta|x)}{\sup_{\theta \in \Theta} L(\theta|x)},$$

where Θ is the set of all possible θ values, and Θ_0 is the null-set.

   (a) (3 points) Explain why $\lambda(x)$ must always fall in the interval $[0, 1]$.

   (b) (2 points) For the Likelihood Ratio test, the rejection region is values of x where the ratio is small: $R = \{x : \lambda(x) \leq c\}$. Conceptually, why do *small* values of $\lambda(x)$ lead us to reject the null hypothesis?

2. A restaurant assigns one chef to work the same shift for an entire week. If Chef 1 is working, the number of customers served per day follows a Poisson distribution with rate $\lambda_1 = 10$. If Chef 2 is working, the daily customers served follows a Poisson distribution with rate λ_2.

You observe the number of customers for the first three days of the week: X_1, X_2, X_3, which we assume are independent draw from either $Pois(\lambda_1)$ or $Pois(\lambda_2)$. You want to test the null hypothesis $H_0 : \lambda = \lambda_1$ (Chef 1 is working) against the alternative $H_1 : \lambda = \lambda_2$ (Chef 2 is working).

Let your test-statistic be the total number of customers observed on the first three nights: $T(X) = \sum_{i=1}^{3} X_i$, and consider the Hypothesis test with rejection region:

$$R = \{X : T(X) > 30\}.$$

   (a) (2 points) Calculate or estimate the Type I error rate (α) of this test. Does the test control the Type I error rate at the traditional $\alpha = 0.05$ level? You can use software to approximate the Poisson probabilities.

   (b) (2 points) In many contexts, controlling the Type I error rate is standard because false positives are "more consequential" than false negatives. Conceptually, why does controlling the Type I error rate make very little sense for this problem?

   (c) (3 points) Consider redesigning this test. Instead of rejecting if $Y > 30$, pick a value c such that we reject when $Y > c$. Rather than picking c to strictly control the Type I error rate (as it doesn't make much sense to do so for this problem), propose an alternative statistical criterion for picking c. State your criterion, briefly explain why it is appropriate for this scenario, and then use it to find an optimal integer value for c.

3. Let $X_1, \ldots, X_n$ be a random sample from an Exponential distribution, with pdf:

$$f_{X_i}(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0.$$

Suppose you want to test $H_0 : \lambda \leq \lambda_0$, against $H_1 : \lambda > \lambda_0$. Let your test statistic be $T(X) = \sum_{i=1}^{n} X_i$.

   (a) (1 point) Is $T(X)$ a sufficient statistic?

   (b) (2 points) Show that the exponential distribution family has the Monotone Likelihood Ratio (MLR) property with respect to $T(X)$. That is, assuming that $\lambda_2 > \lambda_1$, show that the likelihood ratio $L(\lambda_2|X)/L(\lambda_1|X)$ is a monotone function of $T(X)$.

   (c) (2 points) Given that the sum of n independent Exponential(λ) random variables follows a Gamma distribution with shape parameter n and rate λ, describe how you could find a rejection region using a likelihood ratio test with significance level $\alpha = 0.05$ (You don't actually need to calculate the number for the rejection region, just describe mathematically or in words how to find the region. If using words, be precise.)